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The force on a prestressed body in a curved space

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Abstract. Exploring some aspects of the linearized theory of general relativity, we discover new possibilities for experimental tests. A new design for a gravitational wave detector is suggested. The starting point of these investigations is a part of newtonian mechanics whose study helps to understand some of the puzzling features of general relativity. The approximations made are as numerous and as severe as in most similar investigations.

1. Introduction

We study the motion of bodies which are assumed not to influence the metric. This assumption is essential, as I know of no method to solve the general two-body problem of general relativity. On the other hand, the equations which govern the motion in a given space according to general relativity or newtonian mechanics can be solved by numerical methods. This, however, does not provide much physical insight. We therefore proceed along a different path and assume that all spaces involved are almost flat and that coordinates are chosen such that

$$|g_{ik}| \simeq \delta_{ik} \qquad \text{for all } i, k, \tag{1}$$

where g_{ik} is the metric tensor and δ_{ik} is the Kronecker delta. We also assume that the diameter of the test body or test particle is small if compared with the radii of characteristic curvatures of the given space. These assumptions allow us to give physical meaning to a total force acting on the test body. This force has, of course, no tensorial invariance (see Synge 1964 for a discussion of related topics). The manner in which this force is influenced by an initial stress is studied in the present paper. The initial stress of an elastic medium in space–time is considered in a work by Papapetrou (1972).

The spaces considered in §§ 2, 3 and 4 are of dimensions 2, 3 and 4, respectively, with signatures 2, 3 and 2, respectively. Greek suffixes take the values 1, 2, 3, Latin suffixes 1, 2 in § 2 and 1, 2, 3, 4 in § 4 with the summation convention in each case. Partial derivatives with respect to the coordinates are indicated by commas (eg $p_{,i} \equiv \partial p/\partial x_i$). We use real time, $x_4 = t$, and the units are chosen so that the gravitational constant and the speed of light are both unity.

2. Motion of a particle on a surface according to newtonian mechanics

Under no applied force, a particle on a smooth surface with metric $g_{mn}(x_k)$ moves along a geodesic with constant speed. If, however, the particle of unit mass is under the influence of a force with potential $W(x_k)$, then the particle describes a geodesic in the configuration space with metric

$$a_{mn} = (H-W)g_{mn}\,\mathrm{d}x_m\,\mathrm{d}x_n, \tag{2}$$

where H is the constant total energy of the particle. The particle does follow a geodesic but not on the actual surface (Synge and Schild 1966).

We now apply these results to the motion of an elastic membrane on a smooth surface. The potential energy of the membrane (the 'particle') is equal to its elastic strain energy and depends on the intrinsic geometry of the unstrained membrane, and, in addition, on the intrinsic geometry of its location on the surface. Such a particle will move along a straight line in the flat part of a surface; in the non-flat part, however, its trajectory will deviate from a geodesic. A 'particle' with non-zero diameter traces out a strip rather than a line. When we say that the particle does not follow a geodesic, we mean that there does not exist a geodesic lying entirely within the strip.

3. Force on a prestressed solid in a curved three-space

We now adapt the results of § 2 to a curved three-space. We are again dealing with newtonian mechanics—except for the space being curved; but now, instead of using a configuration space as in (2), we try to obtain the force acting on the isotropically elastic solid. This modified approach will facilitate comparison between the results of § 3 and § 4. We assume that the body, brought to rest in flat space, is prestressed or 'self-straining' (Southwell 1969). This means that in euclidean space E_3 the stress tensor $\tau_{\alpha\beta}(z_\nu)$ of the body satisfies

$$\tau_{\alpha\beta,\beta} = 0 \qquad \text{throughout the body,} \tag{3a}$$

$$\tau_{\alpha\beta}n_\beta = 0 \qquad \text{on the boundary,} \tag{3b}$$

but $\tau_{\alpha\beta} \neq 0$ for some α, β. Here n_β is the normal vector to the surface bounding the body and the z_ν are rectangular cartesian coordinates. We think of the coordinate axes as fixed in the body and take the centroid as origin. Equations (3*a*) and (3*b*) express absence of body and surface forces, respectively. Associated with $\tau_{\alpha\beta}$ is the strain tensor $\epsilon_{\alpha\beta}$ given by

$$\epsilon_{\alpha\beta} = \{(1+\sigma)\tau_{\alpha\beta} - \sigma\delta_{\alpha\beta}\tau_{\nu\nu}\}E^{-1} \tag{4}$$

where the constants E and σ are Young's modulus and Poisson's ratio, respectively. We assume that the initial stress $\tau_{\alpha\beta}$, although small, is large if compared with the changes in stress produced by placing this body in the given curved space. This assumption not only makes the calculations much simpler, but also makes the results more interesting. To obtain the stress of the body in this space, we first consider two related problems.

(i) For every simply-connected isotropically elastic body which is self-straining in E_3, there exists a riemannian three-space V_3 in which this body is stressless. If the self-stress satisfying (3) is denoted by $\sigma_{\alpha\beta}$, this space has the metric tensor

$$g_{\alpha\beta} = \delta_{\alpha\beta} + 2\gamma_{\alpha\beta}, \tag{5}$$

where

$$\gamma_{\alpha\beta} = \{(1+\sigma)\sigma_{\alpha\beta} - \sigma\delta_{\alpha\beta}\sigma_{\nu\nu}\}E^{-1}. \tag{6}$$

(ii) The inverse problem—knowing V_3 in which the body is stressless, and finding the self-stress it has in E_3—is slightly more complicated. Having $g_{\alpha\beta}$, we cannot assume $g_{\alpha\beta}$ equals $g_{\alpha\beta}+2\gamma_{\alpha\beta}$; assuming an infinitesimal coordinate transformation

$$x_\alpha \to x_\alpha + \xi_\alpha(x_\beta) \tag{7}$$

has been made, we only know that

$$g_{\alpha\beta} = \delta_{\alpha\beta} + 2\gamma_{\alpha\beta} + \xi_{\alpha,\beta} + \xi_{\beta,\alpha}. \tag{8}$$

Now let us bring the body, whose self-stress in E_3 is $\tau_{\alpha\beta}$, into V_3 of (8) (and let the location be the one in which a body of the same shape but with self-stress $\sigma_{\alpha\beta}$ in E_3 is stressless). The stress $T_{\alpha\beta}$ of the body in this location is not unique because generally the body would move unless constrained. The constraining forces, however, are small under the assumptions made, and we have

$$T_{\alpha\beta}(x_\mu) \simeq \tau_{\alpha\beta}(z_\nu(x_\mu)) - \sigma_{\alpha\beta}(x_\mu), \tag{9}$$

where the components of $\sigma_{\alpha\beta}$ are much smaller than those of $\tau_{\alpha\beta}$. The relation between z_ν and x_μ is linear and corresponds to a translation and a rotation of the body. If there is no rotation or if $\tau_{\alpha\beta}$ is spherically symmetric, we will have

$$x_\mu = z_\mu + X_\mu, \tag{10}$$

where the X_μ are the coordinates of the centroid of the body. Associated with $T_{\alpha\beta}$ is the strain tensor

$$E_{\alpha\beta} = \epsilon_{\alpha\beta} - \gamma_{\alpha\beta}. \tag{11}$$

The total strain energy of an elastic body is defined by (we can ignore $(\det g_{\alpha\beta})^{1/2}$ in the integrand)

$$W(X_\mu, \ldots) = \tfrac{1}{2} \iiint_{\mathscr{D}} E_{\alpha\beta} T_{\alpha\beta}\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3, \tag{12}$$

where the integration is carried out over the domain $\mathscr{D}$ occupied by the body. Generally $T_{\alpha\beta}$, and $E_{\alpha\beta}$ and W, depend not only on the position of the centroid, but also on the orientation of the body. For simplicity, however, let us assume that the orientation is unimportant and, therefore, $W = W(X_\mu)$; this, for instance, is the case if $\tau_{\alpha\beta}$ exhibits spherical symmetry. Since the space is almost flat, we can give physical meaning to a total force F_ν defined by

$$F_\nu(X_\mu) = -\partial W(X_\mu)/\partial X_\nu. \tag{13}$$

From (9) and (11) we have

$$T_{\alpha\beta}E_{\alpha\beta} = \tau_{\alpha\beta}\epsilon_{\alpha\beta} - \tau_{\alpha\beta}\gamma_{\alpha\beta} - \sigma_{\alpha\beta}\epsilon_{\alpha\beta} + \sigma_{\alpha\beta}\gamma_{\alpha\beta}. \tag{14}$$

The contribution to W of the last term is negligible, whereas the first term will give a large, but constant, contribution. Applying (4) and (6), we find

$$\tau_{\alpha\beta}\gamma_{\alpha\beta} = \sigma_{\alpha\beta}\epsilon_{\alpha\beta}. \tag{15}$$

We thus have

$$F_\nu(X_\mu) \simeq \frac{\partial}{\partial X_\nu} \iiint_{\mathscr{D}} \tau_{\alpha\beta}(z_\lambda(x_\rho))\gamma_{\alpha\beta}(x_\rho)\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3. \tag{16a}$$

Using (10) and (8), this transforms to

$$F_\mu(X_\mu) = \frac{\partial}{\partial X_\nu} \iiint\limits_{\mathscr{D}'} \tau_{\alpha\beta}(z_\lambda)\gamma_{\alpha\beta}(z_\phi + X_\phi)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3$$

$$= \iiint\limits_{\mathscr{D}'} \tau_{\alpha\beta}(z_\lambda)\gamma_{\alpha\beta,\nu}(z_\phi + X_\phi)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3$$

$$= \iiint\limits_{\mathscr{D}} \tau_{\alpha\beta}(z_\lambda(x_\rho))\gamma_{\alpha\beta,\nu}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$

$$= \tfrac{1}{2} \iiint\limits_{\mathscr{D}} \tau_{\alpha\beta}(z_\lambda(x_\rho))(g_{\alpha\beta} - \delta_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha})_{,\nu}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3. \tag{16b}$$

Using Green's theorem and (3), we find, finally,

$$F_\nu(X_\mu) = \tfrac{1}{2} \iiint\limits_{\mathscr{D}} \tau_{\alpha\beta}(z_\lambda(x_\rho))g_{\alpha\beta,\nu}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3, \tag{16c}$$

that is, the total force can be found without knowledge of ξ_α.

4. Force on a self-straining body in general relativity

We now consider the same problem from the standpoint of general relativity and assume that, in the almost-flat four-space with metric g_{ik}, there is a body (eg the antenna) which does not influence the metric. We again assume that the body is prestressed and has in flat space a stress tensor $\tau_{\alpha\beta}$ satisfying (3). The matter tensor of the test body, although not satisfying Einstein's equation, is supposed to satisfy the so called equations of motion

$$T^{ik}{}_{|k} \equiv T^{ik}{}_{,k} + \Gamma^i_{mk}T^{mk} + \Gamma^k_{mk}T^{im} = 0, \tag{17}$$

where the Γ^a_{bc} denote Christoffel symbols of the second kind.

We now formulate our problem as follows. Assume we have knowledge of T^{ik} (and thus of the motion) for a test body lacking large initial stress. Suppose we have another test body with the same shape, initial position, initial velocity, etc, except for a large initial stress $\tau_{\alpha\beta}$. We then ask which external force do we have to apply to the second body so that the second body, in spite of the difference in stress, has the same world-tube as the first body? In order to keep the complexity of the representation down, we assume that the velocity of the test body is small. There is, therefore, no need to introduce Lorentz transformations of $\tau_{\alpha\beta}$, and this will facilitate comparison of our results with (16). Let S^{ik} denote the matter tensor of the second body. We have

$$S^{ik} \equiv T^{ik} + \tau_{ik}(x_\alpha, x_4) + f^{ik} \tag{18}$$

with $\tau_{i4} \simeq 0$ and the components of f^{ik} small in comparison with τ_{ik}. Note that f_{ik} is not unique; it will depend on the way the constraining external force is applied. From (18) we find, with

$$T^{ik}{}_{|k} = S^{ik}{}_{|k} = \tau_{ik,k} = 0, \tag{19}$$

that

$$-f^{ik}{}_{|k} = \Gamma^i_{mk}\tau_{mk} + \Gamma^k_{mk}\tau_{im} \simeq -f^{ik}{}_{,k} \equiv -f^i. \tag{20}$$

It is plausible to interpret $-f^i$ as the density of the external force which is required in order to keep the second body on course. We have

$$-f^\nu \simeq \Gamma^\nu_{\alpha\beta}\tau_{\alpha\beta}+\Gamma^c_{\mu c}\tau_{\nu\mu} \simeq \tfrac{1}{2}(g_{\nu\alpha,\beta}+g_{\nu\beta,\alpha}-g_{\alpha\beta,\nu})\tau_{\alpha\beta}+\tfrac{1}{2}(g_{\beta\beta,\mu}-g_{44,\mu})\tau_{\nu\mu} \tag{21a}$$

$$-f^4 \simeq \Gamma^4_{\alpha\beta}\tau_{\alpha\beta} \simeq -\tfrac{1}{2}(g_{\alpha 4,\beta}+g_{\beta 4,\alpha}-g_{\alpha\beta,4})\tau_{\alpha\beta}. \tag{21b}$$

Defining the total external force K_i by

$$K_i \equiv \iiint -f_i \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \equiv \int_{\mathscr{D}} -f_i\,\mathrm{d}V \tag{22}$$

we find, using Green's theorem and (3), that

$$K_i = -\tfrac{1}{2}\int_{\mathscr{D}} g_{\alpha\beta,i}\tau_{\alpha\beta}\,\mathrm{d}V, \tag{23}$$

and, in addition, K_i, but not f_i, is invariant under infinitesimal coordinate transformations $x_i \to x_i + \xi_i(x_j)$.

Comparing (23) with (16) we find that

$$F_\nu = -K_\nu. \tag{24}$$

This means that under the assumption made, newtonian mechanics and general relativity give the same force; the minus sign in (24) results from defining K_i as a reaction. The component K_4, which indicates a changing mass of the test body, is obviously zero for the type of metric assumed in § 3. There we had to restrict ourselves to time-independent metrics in order to be able to define a potential energy. In § 3 we also assumed that test bodies are made of elastic material, whereas no such assumption was made in § 4. We actually could derive (16) for more general materials (for details see Biot 1965).

We obtained $+K_i$ as the external force which we have to apply in order to give the self-strained test body the same world-tube as the unstrained body. But we are likely to assume in most applications that $-K_i$ is equal to a perturbing force acting on a self-straining body and producing minor deviations of its world-tube from the world-tube of the unstrained test body. Self-consistency requires, of course, that changes in motion resulting from K_i turn out to be minor. To get the motion, we treat K_i as a force in flat space, and will therefore use Newton's law of motion or a relativistic analogue.

5. Possible tests

To facilitate estimation of the magnitude of K_ν, we expand $g_{\alpha\beta,\nu}(x_i)$ in (23) in a Taylor series about the point $x_\alpha = 0$, and get thus

$$-2K_\nu = g_{\alpha\beta,\nu|0}\int \tau_{\alpha\beta}\,\mathrm{d}V + g_{\alpha\beta,\nu\mu|0}\int \tau_{\alpha\beta}x_\mu\,\mathrm{d}V + \tfrac{1}{2}g_{\alpha\beta,\nu\mu\partial|0}\int \tau_{\alpha\beta}x_\mu x_\partial\,\mathrm{d}V + \dots. \tag{25}$$

However, it follows from (3) that

$$\int_{\mathscr{D}} \tau_{\alpha\beta}\,\mathrm{d}V = \int_{\mathscr{D}} \tau_{\alpha\beta}x_\mu\,\mathrm{d}V = 0. \tag{26}$$

The first two terms on the right-hand side of (25) are therefore zero and the lowest term in the expansion of K_ν is proportional to the third derivative of $g_{\alpha\beta}$. (This should not come as a surprise—a prestressed membrane on a sphere (R_{1212} = constant; $g_{ab,nm} \neq 0$, $g_{ab,nms} = 0$) will obviously not move; ie $K_\nu = 0$ in this case.) See Appendix 1 for an example. The force between two self-straining bodies resulting from self-strain is therefore almost always much smaller than the force between the two masses of Weber's antenna, (Weber 1960, Madore and Papapetrou 1971) which is proportional to the second derivative of g_{ab}.

Let us now, however, investigate the force f_ν which is acting on a part of a prestressed body. The results of § 3 are easily adapted to this problem. Equations (3)–(11), (14) and (15) remain unchanged. Replacing, in (12), the domain $\mathscr{D}$ by the subdomain $\mathscr{d}$ occupied by the part of the body, will give the strain energy w of this part. Proceeding as in (13)–(16*b*), we find

$$f_\nu = \int_{\mathscr{d}} \tau_{\alpha\beta}\gamma_{\alpha\beta,\nu}\, \mathrm{d}V. \tag{27}$$

There exists, however, no equivalent of (16*c*). We therefore have to find ξ_α in order to find f_ν. An example for this is given in Appendix 2. Equations (26) are no longer true if $\mathscr{D}$ is replaced by $\mathscr{d}$, and therefore f_ν is proportional to the first derivative of $g_{\alpha\beta}$—for a certain coordinate system. (This should not come as a surprise—tidal forces would depend on the first derivative of the newtonian potential if positive and negative masses would exist; but positive and negative stresses do in fact exist.)

Assuming that a similar result is true for a prestressed body in the field of a plane gravitational wave (see (43)) with frequency ω, we find that a single cube of length 100 cm made of titanium alloy with a self-stress close to the ultimate strength is better than Weber's antenna if $\omega < 10^{-6}\ \mathrm{s}^{-1}$. Here we ignored questions of resonance and compared only the 'driving forces'. It is, however, not certain that periodic waves are responsible for the observations made by Weber. Colliding bodies, for instance, produce a wave for which the g_{ik} have approximately the shape of a step function. The time integral over f_ν would then be non-zero, whereas the time integral over forces which depend on $g_{ab,cd}$ would vanish. A prestressed body might therefore be a better antenna for such waves.

Where else might we expect detectable effects of the self-strain? The ratio of initial stress to mass density might be large in elementary particles, their world-line therefore might deviate somewhat from a geodesic. Electrons which are quite large (their Coulomb field included) are prime candidates (Witteborn and Fairbank 1967).

Appendix 1

Suppose the self-straining cube with self-stress

$$\begin{aligned}
\tau_{11} &= -2kx_2x_3(a^2-x_1^2)^2 \\
\tau_{12} &= 2kx_1x_3(a^2-x_1^2)(a^2-x_2^2) \\
\tau_{13} &= 2kx_1x_2(a^2-x_1^2)(a^2-x_3^2) \\
\tau_{23} &= k(a^2-3x_1^2)(a^2-x_2^2)(a^2-x_3^2)
\end{aligned} \tag{A.1}$$

and boundary

$$x_1, x_2, x_3 = \pm a \tag{A.2}$$

is in V_4 with metric (a plane wave solution of Einstein's linearized equations)

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 + 2\,dx_3\,dx_4 A\cos\omega(x_1 - x_4). \tag{A.3}$$

From (23) we find

$$K_2 = K_3 = 0,$$

$$K_1 = -\tfrac{1}{2}\iiint_{-a}^{+a} g_{23,1}\tau_{23}\,dx_1\,dx_2\,dx_3 = -K_4$$

$$= -\tfrac{32}{9}\omega^{-2}Aka^6\sin(\omega x_4)\{(3 - a^2\omega^2)\sin\omega a - 3a\omega\cos\omega a\}. \tag{A.4}$$

We get, for $\omega a \ll 1$,

$$K_1 = -\tfrac{32}{135}\omega^3 Aka^{11}\sin\omega x_4 + O(\omega^5). \tag{A.5}$$

Appendix 2

Let the metric of a V_3 be given by

$$g_{11} = 1 + \tfrac{1}{3}R(x_3^2 - 2x_2^2);\qquad g_{22} = 1 + \tfrac{1}{3}R(x_3^2 - 2x_1^2);\qquad g_{33} = 1 + \tfrac{1}{3}R(x_1^2 + x_2^2);$$

$$g_{12} = \tfrac{2}{3}Rx_1x_2;\qquad g_{13} = -\tfrac{1}{3}Rx_1x_3;\qquad g_{23} = -\tfrac{1}{3}Rx_2x_3. \tag{A.6}$$

Given that an elastic sphere of radius A is stressless in this V_3 if its centre has coordinates $x_\alpha = 0$, find the strain $\gamma_{\alpha\beta}$ of this sphere in E_3. It follows from axial symmetry of $g_{\alpha\beta}$ that

$$\begin{aligned} \xi_1 &= x_1 S(r, x_3) \\ \xi_2 &= x_2 S(r, x_3) \\ \xi_3 &= T(r, x_3). \end{aligned} \tag{A.7}$$

where $r^2 = x_1^2 + x_2^2$. The functions S and T are found from the following system of equations:

$$\begin{aligned} 2\gamma_{\alpha\beta} &= g_{\alpha\beta} - \delta_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \\ \sigma_{\alpha\beta} &= \frac{E}{1+\sigma}\left(\frac{\sigma}{1-2\sigma}\delta_{\alpha\beta}\gamma_{\nu\nu} + \gamma_{\alpha\beta}\right) \\ \sigma_{\alpha\beta,\beta} &= 0 \\ \sigma_{\alpha\beta}n_\beta &= 0 \qquad \text{for } r^2 + x_3^2 = A^2. \end{aligned} \tag{A.8}$$

A lengthy calculation gives

$$\begin{aligned} S &= R(114B + 42)^{-1}\{3A^2(3B+1) - (22B+4)x_3^2 - (B+1)r^2\} \\ T &= R(114B + 42)^{-1}\{-6A^2(3B+1)x_3 + (2B+2)x_3^3 + (23B+5)x_3r^2\}, \end{aligned} \tag{A.9}$$

where $B = \sigma/(1 - 2\sigma)$. The strain is thus found.

Metric (A.6) was obtained from the linearized Riemann tensor

$$P_{1212} = -2P_{1313} = -P_{2323} = 2R \equiv 2md^{-3} = \text{constant}, \tag{A.10}$$

by use of the formula

$$g_{\alpha\beta} = \gamma_{\alpha\beta} + 2x_\gamma x_\delta \int_0^1 P_{\alpha\gamma\delta\beta}(x_\nu\lambda)\lambda(1-\lambda)\,\mathrm{d}\lambda. \tag{A.11}$$

These values for $P_{\alpha\beta\gamma\delta}$ in turn were chosen because they coincide at $x_\alpha = 0$ with the linearized Riemann tensor calculated for the space-part of the linearized Schwarzschild metric; namely

$$g_{\alpha\beta} = \delta_{\alpha\beta} + 2m\{r^2 + (x_3 + d)^2\}^{-1/2}. \tag{A.12}$$

The $\sigma_{\alpha\beta}$ of (A.8) might therefore describe the changes in the moon's self-stress which are produced by moving the moon from an earth–moon distance d to flat space.

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